Vortex distribution in neutron stars: gravitational effects.

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Abstract

Neutron stars are supposed to be mainly formed by a neutron superfluid. The angular momentum is given by the vortex array within the fluid, and a good account of the observable effects is determined by its coupling with the crust. In this article we show that the gravitational field introduces important modifications in the vortex distribution and shape. The inertial frame dragging on the quantum fluid produces a decrease in the vortex density, which for realistic models is in the order of 15%. This effect is relevant for neutron star rotation models and can provide a good framework for checking the quantum effect of the frame dragging.

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I. INTRODUCTION

In most quantum systems the gravitational effects are extremely small, but there are some remarkable exceptions. One which has been already extensively studied is the generation of inhomogeneities in the inflationary phase of the universe, through the amplification and the transition to a classical regime of the quantum fluctuation of the inflaton field [1]. In this case the relevance of the phenomena is related to the propagation of modes with a wavelength comparable to the radius of the horizon of the universe, i.e., the gravitational and the quantum characteristic lengths are of the same order. A natural question which arises is if there are other cases accessible from a phenomenological point of view, and the best candidate for an affirmative answer is provided by the neutron stars. There is a consensus not only about their existence but also about their direct relation with pulsars [2]. As it is well known, the pulsar dynamics shows a very interesting phenomenology closely related to neutron star models. Among the most suggestive phenomena displayed by pulsars are the glitches, which are closely linked to the internal dynamics of the star. The post-glitch relaxation of the angular velocity has a large time scale, which is interpreted as evidence of a crust containing a superfluid medium [3].The models usually assume a solid exterior crust, which is the directly observable zone, and inner regions mostly constituted by superfluid neutrons. The superfluid phase would be stable at the very high pressure in the interior of the star [4]. This image is supported by theoretical computations [5] as well as observational evidence [6]. Given that a superfluid flow is irrotational, the angular momentum of the star must be supported by vortex lines. They are coupled to the exterior crust, and the different phenomena associated to the glitches are related with this coupling [7]. These models give a qualitatively correct description for the glitches dynamics. In this article we show that gravitational effects could be a very relevant ingredient to this description. Although this article is concerned with neutron stars, our conclusions can be applied to the rotation of boson stars.

In the following section we give a brief introduction to the subject of the rotation of a

superfluid star, using the weak field approximation for the gravitational field. This approximation is useful for understanding the phenomena, but it is not suitable for studying effects that involve the gravitational field in the interior of the neutron stars because of their high mass density. For this reason in Section III we analyze the rotation of a superfluid star using a different and more adequate approach. We consider there the exact expression for the gravitational interaction, and expand the metric in powers of the angular velocity. This is a small parameter for a neutron star, and thus we can restrict the expansion to linear tems without a significant error. Finally, in Section IV we study in detail the vortex distribution in the star and the effects of the gravitational field on this distribution using a covariant description for the superfluid.

II. WEAK FIELD APPROXIMATION TO THE SUPERFLUID STAR ROTATION

To get an insight into the physical aspects of this problem we will discuss it in the first place using a weak field approximation. Thus we will assume a post-Newtonian metric [8]:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad , \quad h_{\mu\nu} \ll 1 \,, \tag{1}$$

where the gravitational field is described by a vectorial potential, $\vec{h}_i = h_{0i}$, and a scalar one, $\phi = \frac{h_{00}}{2}$. In this case the Hamiltonian for a non relativistic particle becomes [9]:

$$H = (1 + 3\,\phi) \frac{(\vec{p} + m\,\vec{h})^2}{2m} + m\,\phi \,. \tag{2}$$

Except for the factor $(1 + 3\,\phi)$, which gives the red shift effect, there is a clear similarity between this Hamiltonian an the corresponding one for a particle in an electromagnetic field, where the electric charge e should be replaced by the mass m, the vector electromagnetic potential $\vec{A}$ by $-\vec{h}$, and the electric potential by the Newtonian one. Extending the analogy one could suppose that a gravitational Meissner effect, the expulsion of the field $\nabla \times \vec{h}$, is present among the gravitational phenomena that take place in a superfluid star, as was proposed in Ref. [10]. However, we can argue that this is not the case on the basis of the

Mach principle. It tells us that the field source tries to impose its own rest frame to the other bodies through the gravitational interaction. This implies that the gravitational effect on the bodies has a paramagnetic nature independently of the substances they are made of, and thus there is no such Meissner effect. This point will be formalized later.

Let us now analyze the slow rotating star in this approximation. The energy-momentum tensor of the fluid is $T^{\mu\nu} = (p + \rho)u^\mu u^\nu + pg^{\mu\nu}$, where p is the pressure and ρ the energy density. Expanding this tensor up to first order in the velocity and using the weak field approximation to the Einstein equations in the harmonic gauge, the equations for the field $\vec{h}$ become

$$\Delta \vec{h} = 16\pi G \left[(\rho + p)\,\vec{v} + \frac{1}{2}(3\rho + p)\,\vec{h} \right] \qquad , \tag{3}$$

$$\vec{\nabla}.\vec{h} = 0. \tag{4}$$

Based on the analogy with a superconductor, the Hamiltonian (2) implies that $\vec{\nabla} \times (\vec{v} + \vec{h}) = 0$. Furthermore, in a superfluid star we have a stationary rotating fluid where $\vec{\nabla}.\vec{v} = 0$. These two conditions, together with Eq.(4) give $\vec{v} = -\vec{h}$. Thus Eq.(3) leads to

$$\Delta \vec{h} = -8\pi G \,(\rho - p)\,\vec{h}. \tag{5}$$

As $\rho > p$, the paramagnetic character mentioned above is clearly stated. A similar result was obtained in Ref. [11], but there is a difference because the authors do not take into account the term proportional to $\vec{h}$ on the right hand side of Eq.(3). In the present context this term has the same weight as the term proportional to $\vec{v}$, since $\vec{v} = -\vec{h}$, and it should not be neglected.

The study of the rotation of the star reduces to the analysis of Eq.(5), which formally can be considered as a Schrödinger equation for $\vec{h}$ with a potential well proportional to $\rho - p$. In absence of vortices the phases induced in the superfluid by the rotation and the field $\vec{h}$ cancel exactly, and a rotational state of the star is given by a bounded regular solution of this equation that nullifies $\vec{h}$ at infinity. In general the existence of this solution implies a deep or long potential well that leads to an unstable star that will collapse to a black hole.

At least this result shows that the weak field approximation is not suitable for analyzing the possibility of a vortexless rotational state, because non linear effects are important.

III. SUPERFLUID STAR ROTATION

In this section we will analyze the superfluid star rotation on the basis of an expansion of the metric tensor $g_{\varphi t}$ in powers of the angular velocity Ω, which in the case of neutron stars can be considered as a small parameter. This last point allows us to restrict the expansion up to linear terms in Ω [12]. At first order on the angular velocity the metric takes the form

$$ds^2 = -e^{2\Phi}dt^2 + e^{2\Lambda}dr^2 + r^2(d\theta^2 + \sin^2\theta \; d\varphi^2) - 2r^2\sin^2\theta \; \omega \, d\varphi \, dt \,, \tag{6}$$

where $\omega(r,\theta)$ represents the angular velocity that takes a free falling object from infinity to the point (r,θ), and which corresponds to the local inertial frames rotation with respect to the fixed stars. The potentials Φ and Λ are even functions of the angular velocity, and therefore in this approximation they are the non-perturbed functions of r that can be computed directly from the non-rotating stellar model. For stars at the end of the thermonuclear evolution they can be determined from the equation of state and the central pressure. The equations in this case are:

$$m(r) = \int_0^r 4\pi\rho dr \,, \tag{7}$$

$$\frac{dp}{dr} = -\frac{(\rho + p)\,(m + 4\pi r^3 p)}{r(r - 2m)} \,, \tag{8}$$

$$\frac{d\phi}{dr} = \frac{m + 4\pi r^3 p}{r(r - 2m)} \,, \tag{9}$$

$$e^{2\Lambda} = \frac{r}{r - 2m} \,. \tag{10}$$

The remaining component of the metric, ω, is given by the Einstein equation corresponding to the component $R_{\varphi t}$ of the Ricci tensor. To reduce this equation to a simple and suitable form, we can introduce the angular velocity of the system, which can be precisely characterized by $\Omega = u^\varphi/u^t$ in terms of the fluid velocity u^ν. This magnitude Ω represents the angular velocity measured by an observer at rest with respect to the fluid. The minimal

energy configuration, and in consequence the stable one, has Ω constant, which restricts the problem to the case of a uniform rotation. If we assume that the constitutive matter behaves as a perfect fluid, $T_{\varphi t}$ is:

$$T_{\varphi t} = -r^2 \sin{}^2\theta \left((\rho + p) \, \Omega - \rho\omega \right) . \tag{11}$$

Besides, using the azimuthal symmetry of the rotating star, we can expand ω as follows:

$$\omega(r,\theta) = \sum_{l=1}^{\infty} \omega^l(r) \left(-\frac{1}{\sin\theta} \frac{dP_l}{d\theta} \right) , \tag{12}$$

where P_l is the Legendre polynomial of degree l. With these ingredients the perturbed Einstein equation becomes

$$\omega^l_{,rr} + (\frac{4}{r} - \Lambda' - \Phi') \, \omega^l_{,r} + \frac{2}{r} \, (\frac{1}{r} + \Phi' - \Lambda' - \frac{l(l+1)}{2r} e^{2\Lambda}) \, \omega^l$$
$$= 16\pi e^{2\Lambda} (\frac{1}{2}(\rho + 3p) \, \omega^l - (\rho + p) \, \Omega \, \delta^l_1) . \tag{13}$$

The only information which remains to be introduced refers to the superfluid state of the star matter. The superfluid is characterized by the curved space-time covariant generalization of $\nabla \times \, \bar{v} = 0$, which is satisfied by a superfluid in absence of vortices in a flat space-time. According to this the quadrivelocity must satisfy

$$\epsilon^{ijkl} \xi^t_l (u_{j,k} - u_{k,j}) = 0 , \tag{14}$$

where ξ is a time-like Killing vector. Due to the symmetries of the system this relation reduces to $u_{\varphi,r} = 0$, which we can rewrite as

$$\frac{d}{dr}(u^\varphi g_{\varphi\varphi} + u^t g_{t\varphi}) = 0 . \tag{15}$$

Substituting the metric components by the expressions which correspond to (6), we obtain that the quantity $r^2(u^\varphi - u^t\omega)$ is independent of r and therefore null. Hence we have

$$\frac{u^\varphi}{u^t} = \Omega = \omega . \tag{16}$$

We will reconsider this relation further on to include the presence of vortices. By introducing it now in Eq.(13) and using (7) we obtain the differential equation satisfied by ω in the superfluid star:

$$\omega^l{}_{,rr} + (\frac{4}{r} - \Lambda' - \Phi')\,\omega^l{}_{,r} + \frac{2 - l(l+1)}{r^2}e^{2\Lambda}\,\omega^l = 0\,. \tag{17}$$

In this equation we can substitute $\omega = \frac{-h^\varphi}{r\sin\theta}$, and develop up to the linear terms in the fields, thus recovering Eq.(5). For Λ and Φ regular at the origin, i.e., nonsingular stars, the only solution with regular geometry is $\omega = 0$. Therefore there are no solutions for rotating superfluid stars without vortices. A similar result was obtained in Ref. [13] when analyzing boson stars. In this case there is not an effective equation of state and in that paper the boson star structure was computed using the Klein-Gordon equation. Besides, the energy momentum tensor is not isotropic because the radial and tangential pressures are not equal in the general case. However, as in Eq.(13), only the radial pressure is relevant and the resulting equation is again Eq.(17). Due to the scalar field coherence the equation $\Omega = \omega$ is satisfied, and hence the energy momentum tensor is a function of ω only.

IV. ROTATION IN PRESENCE OF VORTICES

Because the superfluid star rotation cannot be achieved as a perturbation of its fundamental state, we are going to study the rotation in presence of vortices. In order to consider the vortex contribution to the superfluid star dynamics in curved space, we will follow an approach analogous to the one proposed by Weinberg [14]. The neutron (and proton) field has a U(1) global symmetry:

$$\Psi(x) \longrightarrow e^{i\Lambda}\,\Psi(x)\,, \tag{18}$$

that leads to the baryonic number conservation. The superfluidity phenomena is related to the formation of a condensate that spontaneously breaks this global symmetry. This is similar to the superconductivity effect, where there is a spontaneous breaking of the U(1) electromagnetic gauge symmetry to Z_2. Based on the physical picture for the neutron star matter, where neutron pairs have nonvanishing expectation values, we will assume that the U(1) baryonic symmetry is broken to Z_2, the subgroup of transformations with $\Lambda = 0$ and

$\Lambda = \pi$. The spontaneous breaking of the symmetry leads to the existence of a Nambu-Goldstone excitation with zero energy in the limit of vanishing momentum. The group transformation acting on the Nambu-Goldstone boson is:

$$\phi(x) \longrightarrow \phi(x) + \Lambda \, . \tag{19}$$

As ϕ parametrizes $\mathrm{U}(1)/Z_2$, ϕ and $\phi + \pi$ are taken to be equivalent. The U(1) invariant density Lagrangian is a function of the derivatives of ϕ and the U(1) fixed neutron fields $\tilde{\Psi}$, and the baryonic Noether current is given by $j^\alpha = \frac{\delta L}{\delta(\partial_\alpha \phi)}$. The most general density Lagrangian allowed by the symmetries is a nonlocal function of the field, but the nonlocality extends over a range of the order of the penetration length of the superfluid. Given that we are interested in the macroscopic fluid motion, we will only consider the local terms in the density Lagrangian that effectively describe the long range behavior. Such terms must be scalars and should be constructed as a contraction of covariant quantities. The only possible factors are the gradient of ϕ, the metric (not the curvature tensor because of the equivalence principle) and a number of fixed tensors that characterize the field $\tilde{\Psi}$ and must satisfy the requirements of spherical symmetry, since they are determined by the unperturbed star. Let us call them $\lambda, \lambda_\mu^{(1)}, \lambda_{\mu\nu}^{(2)}, ...$ As was argued in Ref. [14] the existence of an equilibrium configuration with vanishing ϕ gradients rules out the linear terms in its derivatives . The quadratic terms must not vanish since the system has a spontaneous symmetry breaking. Therefore, the density Lagrangian can be expanded as:

$$\mathcal{L} = \frac{1}{2}(f(\lambda)\,\phi_{,\nu}\phi^{,\nu} + g(\lambda)\,\lambda_{\mu\nu}^{(2)}\,\phi^{,\nu}\phi^{,\mu} + ...) \, . \tag{20}$$

We only need the quadratic terms because $\phi_{,\nu}$ is zero for the static star, and therefore it is a first order quantity of the angular velocity for a rotating one. Hence we have $j_\mu = f(\lambda)\,\phi_{,\mu} + g(\lambda)\,\lambda^{(2)\,\nu}_{\ \ \mu}\,\phi_{,\nu}$. Due to the spherical symmetry the tensor $\lambda_{\mu\nu}^{(2)}$ has the same angular dependence as the metric. Thus $\lambda^{(2)\,\varphi}_{\ \ \varphi}$ is a function of r and we can write $j_\varphi = n(r)\,\phi_{,\varphi} = n(r)\,\partial_\varphi\phi$. The current component j_φ can also be written for small velocities as $n_0(r)\,u_\varphi$, where $n_0(r) = \frac{1}{\sqrt{\gamma}}\frac{dn}{dV}$ is the baryon number density in the fluid rest frame and γ is the determinant of the spacial metric tensor [15]. From here we obtain

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$$u_\varphi = \frac{n(r)}{n_0(r)} \partial_\varphi \phi = \frac{2}{m^*(r)} \partial_\varphi \phi \,, \tag{21}$$

where $m^* = n_0/n$ is a scalar quantity of dimension one that depends only on the fluid conditions. This parameter can be interpreted as the effective mass of the quasiparticles from the equivalence principle, and for a neutron star it takes a value of order of the neutron mass [3] [16] [17]. The Nambu-Goldstone field ϕ should be compared with the double the phase of the Ginzburg-Landau wave function, which is the origin of the factor two in the preceding equation [14].

On the other hand we have

$$\Omega = \frac{u^\varphi}{u^t} = \frac{u_\varphi g^{\varphi\varphi} + u_t g^{\varphi t}}{u^t} \,. \tag{22}$$

This expression becomes

$$\Omega = \frac{u_\varphi}{r^2 \sin^2\theta \, u^t} + \omega = \frac{2 \, \partial_\varphi \phi}{m^* \, r^2 \sin^2\theta} \, e^\Phi + \omega \,. \tag{23}$$

when we keep up to the first order terms in ω. The factor e^Φ gives the redshift of the effective mass. From the rotation symmetry, the periodicity and the continuity of the ϕ field outside singularities we obtain $\phi = n(r,\theta) \, \varphi/2$, where $n(r,\theta)$ is the sum of the indexes of the field singularities held by a circular closed path in a plane orthogonal to the rotation axis, centered on this axis, and containing the point (r,θ). The $n(r,\theta)$ function is also equal to the number of vortices surrounded by the closed path, because each vortex has a topological number one to minimize the energy. In consequence

$$\Omega = \frac{n(r,\theta)}{m^* \, r^2 \sin^2\theta} \, e^\Phi + \omega \,. \tag{24}$$

This equation generalizes the expression (16) by including the presence of vortices. It also generalizes the formula describing the angular velocity and vortex density relation in flat space. From this point of view the new ingredient is the second term on the left hand side of the equation, which corresponds to the dragging of the inertial frames due to the gravitational field. This relation shows that the phase introduced by the vortices in the

superfluid is equal to the sum of the kinetic phase plus a gravitational phase. This phase is due to the metric that makes the covariant and contravariant components different, unlike the electromagnetic case and what can be interpreted from the weak field approximation where it is introduced by a connection.

Given the angular velocity Ω and the star structure at rest, we can state the vortex distribution in a superfluid star. First we compute ω from Eq.(13), with the reasonable supposition that the vortex interaction energy is negligible with respect to the rotation energy. Once ω is obtained we can compute the vortex distribution and shape from Eq.(24). The resulting vortex distribution minimizes the total energy of the fluid, leading to a macroscopic motion equal to the corresponding one without vortices. Since ω is a decreasing function of the radius, the decrease in the vortex density with respect to a flat space is greater at the axis of the star. Besides this, the gravitational field produces a line vortex diffraction in such a way that these lines are not parallel to the rotation axis, as in a flat space, except on the equatorial plane. The angle β between the rotation axis of the star and the direction of the vortex lines at the point (r, θ) is given by

$$\sin \beta = \frac{1}{2} \frac{\kappa(r) \, \sin(2\theta)}{\left(1 + \kappa(r)(\kappa(r) - 2) \, \sin^2(\theta)\right)^{1/2}} \, , \tag{25}$$

where $\kappa(r) = \frac{r}{2} \left(\frac{1}{\Omega - \omega} \frac{d\omega}{dr} - \frac{1}{m*} \frac{dm*}{dr} + \frac{d\Phi}{dr} \right)$.

Outside the star we have $\omega(r) = \frac{2 G J}{r^3}$, where J is the total angular momentum. From here and applying Eq.(24) with $\theta = \frac{\pi}{2}$ and $r = R$, we get the total number of vortices

$$N = m^*(R) \left(\Omega R^2 - \frac{2 G J}{R^3} \right) e^{-\Phi(R)} \, . \tag{26}$$

Even if we do not know the star mass distribution, which is model dependent, it is possible to give an upper bound for the average number of vortices ν_g per invariant area unit, $dA = 2\pi r e^\Lambda dr$. Given that $e^{\Lambda(r)} > e^{-\Phi(R)}$ for $r < R$. this upper bound is:

$$\nu_g = \frac{N}{A} = m^* \, (\Omega - \frac{2 G J}{R^3}) \frac{R^2 e^{-\Phi}}{A} \leq \frac{m^*}{\pi} \, (\Omega - \frac{2 G J}{R^3}) \, . \tag{27}$$

Thus the relative decrease of the average vortex number density with respect to the one at a flat space, $\nu_0 = \frac{m^* \Omega}{\pi}$, is:

$$\Delta = \frac{\nu_0 - \nu_g}{\nu_0} \geq \frac{2\,G\,I}{c^2 R^3} \,, \tag{28}$$

where I is the star moment of inertia. To clarify the meaning of the formula we can put $I = M\,\bar{r}^2$, with $\bar{r}$ the radius of gyration, and thus we get the decrease rate bound as $(\frac{R_s}{R}).(\frac{\bar{r}^2}{R^2})$, where $R_s = \frac{2\,G\,M}{c^2}$ is the Schwarzchild radius of the star. For different neutron star models quoted in Ref.([17]), this lower bound varies from 11% to 15%. The vortex density decrease is greater as the star is more relativistic. These results are interesting at least for two reasons. On the one hand, they can be a significant ingredient for analyzing the dynamics of neutron stars, provided the development of the high pressure nuclear matter theory and pulsar models are accurate enough for describing observations. On the other hand, they provide a framework for the verification of the equivalence principle at the level of quantum systems by checking the effect of frame dragging.

V. ACKNOWLEDGMENTS

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